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Conservation laws of the Hirota-Maxwell-Bloch system and its reductions

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Abstract. It is known that conservation law plays an important role in the study of nonlinear evolution equations and namely to integrability and constants of motion. In this paper, we construct infinitely many conservation laws for the Hirota-Maxwell-Bloch system and its reductions with symbolic computation from the Riccati form of the Lax pair.

1. Introduction

Conservation laws appear in various areas of the applied sciences, such as quantum physics, electromagnetism, plasma physics, physical chemistry, nonlinear optics [1-4]. For a nonlinear equation existence of the infinitely many conservation laws has been claimed to be a definition of the complete integrability [5, 6]. There are various methods [7] to compute conservation laws of nonlinear partial differential equations. A prevalent approach depends on the link between symmetries and conservation laws as stated in Noethers theorem [8-10].

In this work, we construct infinitely many conservation laws for the (1+1)-dimensional Hirota-Maxwell-Bloch system (HMBS) and its reductions with symbolic computation from the Riccati form of the Lax pair. The method that used in this paper was applied to several nonlinear evolution equations in mathematical physics [11-15]. Hirota-Maxwell-Bloch system and its reductions were studied in one and two dimensions [16-23]. Authors have found different kind of solutions but research regarding on infinitely many conservation laws of HMBS has not been presented in detail. Motivated by this reason we find the conservation laws for the (1+1)-dimensional Hirota-Maxwell-Bloch system, the (1 + 1)-dimensional Schrodinger-Maxwell-Bloch system (SMBS) and the (1 + 1)-dimensional complex modified Korteweg de Vries-Maxwell-Bloch equations (cmKdVMB).

The paper is organized as follows. In Section 2, we present the (1+1)-dimensional HMBS and its reductions. In Section 3, we construct infinitely many conservation laws for the (1+1)-dimensional HMBS, SMBS, cmKdVMB. Conclusion is given in Section 4.

2. The Hirota-Maxwell-Bloch system and its reductions

The (1+1)-dimensional Hirota-Maxwell-Bloch system reads as [17]

$$iq_t + \alpha(q_{xx} + 2\delta|q|^2 q) + i\beta(q_{xxx} + 6\delta|q|^2 q_x) - 2ip = 0, \tag{1}$$
$$p_x - 2i\omega p - 2\eta q = 0, \tag{2}$$
$$\eta_x + \delta(q^* p + p^* q) = 0, \tag{3}$$

where q, p are complex functions, η is real function, $\alpha, \beta, \delta, \omega$ are real constants and $\delta = \pm 1$. The symbol $*$ denotes the complex conjugate. The system of equations (1)-(3) admits the following integrable reductions:

2.1. *Case 1:* $\alpha = 1, \beta = 0$

The $(1 + 1)$-dimensional Schrodinger-Maxwell-Bloch equations result when $\alpha = 1, \beta = 0$ [17]:

$$iq_t + q_{xx} + 2\delta|q|^2 q - 2ip = 0, \tag{4}$$
$$p_x - 2i\omega p - 2\eta q = 0, \tag{5}$$
$$\eta_x + \delta(q^* p + p^* q) = 0. \tag{6}$$

2.2. *Case 2:* $\alpha = 0, \beta = 1$

The $(1 + 1)$-dimensional complex modified Korteweg de Vriese-Maxwell-Bloch equations are obtained for the choice $\alpha = 0, \beta = 1$ [17]:

$$iq_t + i(q_{xxx} + 6\delta|q|^2 q_x) - 2ip = 0, \tag{7}$$
$$p_x - 2i\omega p - 2\eta q = 0, \tag{8}$$
$$\eta_x + \delta(q^* p + p^* q) = 0. \tag{9}$$

2.3. *Case 3:* $\alpha = 1, \beta = 1, p = 0, \eta = 0$

The $(1 + 1)$-dimensional Hirota equations are obtained for the choice $\alpha = 1, \beta = 1, p = 0, \eta = 0$ [17]:

$$iq_t + (q_{xx} + 2\delta|q|^2 q) + i(q_{xxx} + 6\delta|q|^2 q_x) = 0. \tag{10}$$

3. Lax pairs and conservation laws

With the Ablowitz-Kaup-Newell-Segur scheme [1], the Lax pair associated with (1)-(3) can be derived as

$$\Psi_x = A\Psi, \tag{11}$$
$$\Psi_t = ((2\alpha\lambda + 4\beta\lambda^2)A + B)\Psi, \tag{12}$$

where

$$\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix},$$
$$A = -i\lambda\sigma_3 + A_0,$$
$$B = \lambda B_1 + B_0 + \frac{i}{\lambda + \omega} B_{-1},$$

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with

$$\begin{aligned}
B_1 &= 2i\beta\delta|q|^2\sigma_3 + 2i\beta\sigma_3 A_{0x},\\
A_0 &= \begin{pmatrix} 0 & q \\ -r & 0 \end{pmatrix},\\
B_0 &= (i\alpha\delta|q|^2 + \beta\delta(q^*q - q^*q_x))\sigma_3 + B_{01},\\
B_{01} &= \begin{pmatrix} 0 & i\alpha q_x - \beta q_{xx} - 2\beta\delta|q|^2 q \\ i\alpha r_x + \beta r_{xx} + 2\beta\delta|q|^2 r & 0 \end{pmatrix},\\
B_{-1} &= \begin{pmatrix} \eta & -p \\ -k & -\eta \end{pmatrix},\\
\sigma_3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\end{aligned}$$

and $r = \delta q^*, \quad k = \delta p^*$, where $\delta = \pm 1$. The compatible condition of system (11)-(12) is

$$A_t - B_x + [A; B] + (2\alpha\lambda + 4\beta\lambda^2)A_x = 0,$$

by direct calculation of above equation, we can yield the (1+1)-dimensional HMBS (1)-(3).

Making use of Lax pair (11)-(12), the infinetely many conservation laws for (1)-(3) could be derived. Introducing the function $\Gamma = \psi_2/\psi_1$ [24], we get the following Ricatti equation via Lax pair (11)-(12):

$$\Gamma_x = -r + 2i\lambda\Gamma - q\Gamma^2. \tag{13}$$

Expanding

$$\Gamma = \sum_{j=1}^{\infty} g_j \lambda^{-j} q^{-1} \tag{14}$$

in (13) and equating the same powers of λ to zero, we have

$$g_1 = -\frac{i}{2}\delta|q|^2, \tag{15}$$

$$g_2 = -\frac{i}{2}(g_{1,x} - \frac{q_x}{q}g_1), \tag{16}$$

$$g_3 = -\frac{i}{2}(g_{2,x} - \frac{q_x}{q}g_2 - g_1^2), \tag{17}$$

$$\ldots \tag{18}$$

$$g_{j+1} = -\frac{i}{2}(g_{j,x} - \frac{q_x}{q}g_j - \sum_{k=1}^{j-1} g_k g_{j-k}), (j = 3, 4, 5...), \tag{19}$$

where g_1, g_2 and g_n are the functions of x and t.

From Lax pair (11)-(12), we have

$$\frac{\psi_{1,x}}{\psi_1} = -i\lambda + q\Gamma, \tag{20}$$

$$\frac{\psi_{1,t}}{\psi_1} = (2\alpha\lambda + 4\beta\lambda^2)a_{11} + b_{11} + ((2\alpha\lambda + 4\beta\lambda^2)a_{12} + b_{12})\Gamma. \tag{21}$$

Taking (20)-(21) into the compability condition $(ln\psi_1)_{xt} = (ln\psi_1)_{tx}$, we get

$$[-i\lambda + q\Gamma]_t = [(2\alpha\lambda + 4\beta\lambda^2)a_{11} + b_{11} + ((2\alpha\lambda + 4\beta\lambda^2)a_{12} + b_{12})\Gamma]_x, \tag{22}$$

where

$$\begin{aligned} a_{11} &= -i\lambda, \\ a_{12} &= q, \\ b_{11} &= \lambda i\beta\delta|q|^2 + i\alpha\delta|q|^2 + \beta\delta(q_x^* q - q^* q_x) + \frac{i}{\lambda+\omega}\eta, \\ b_{12} &= \lambda 2i\beta q_x + i\alpha q_x - \beta(q_{xx} + 2\delta|q|^2 q) - \frac{i}{\lambda+\omega}p. \end{aligned}$$

Substituting (14) with (15)-(19) into (22), and collecting the coefficients of the same power of λ after multiplying both sides of (22) by $\lambda + \omega$, we obtain the infinitely many conservation laws for the equations (1)-(3)

$$\frac{\partial \rho_k}{\partial t} = \frac{\partial J_k}{\partial x} \tag{23}$$

with

$$\begin{aligned} \rho_1 &= -\frac{i}{2}\delta|q|^2, && (24) \\ \rho_2 &= -\frac{\delta}{2}(\frac{1}{2}q_x^* q + i\omega|q|^2), && (25) \\ \rho_3 &= g_3 + \omega g_2, && (26) \\ \cdots & && (27) \\ \rho_n &= g_n + \omega g_{n-1}, \quad (n = 4, 5, 6...) && (28) \end{aligned}$$

and

$$\begin{aligned} J_1 &= (b_{11}^0\omega + i\eta) + (2\alpha\omega + 2i\beta\omega\frac{q_x}{q} + b_{12}^0)g_1 + (2\alpha + 4\beta\omega + 2i\beta\frac{q_x}{q})g_2, && (29) \\ J_2 &= (b_{12}^0\omega\frac{1}{q} - \frac{ip}{q})g_1 + (2\alpha\omega + 2i\beta\omega\frac{q_x}{q} + b_{12}^0)g_2 + (2\alpha + 4\beta\omega + 2i\beta\frac{q_x}{q})g_3, && (30) \\ J_3 &= (b_{12}^0\omega\frac{1}{q} - \frac{ip}{q})g_2 + (2\alpha\omega + 2i\beta\omega\frac{q_x}{q} + b_{12}^0)g_3 + (2\alpha + 4\beta\omega + 2i\beta\frac{q_x}{q})g_4, && (31) \\ \cdots & && (32) \\ J_n &= (b_{12}^0\omega\frac{1}{q} - \frac{ip}{q})g_{n-1} + (2\alpha\omega + 2i\beta\omega\frac{q_x}{q} + b_{12}^0)g_n + (2\alpha + 4\beta\omega + 2i\beta\frac{q_x}{q})g_{n+1}, && (33) \end{aligned}$$

where

$$\begin{aligned} b_{11}^0 &= i\alpha\delta|q|^2 + \beta\delta(q_x^* q - q^* q_x), \\ b_{12}^0 &= i\alpha q_x - \beta(q_{xx} + 2\delta|q|^2 q). \end{aligned}$$

Here ρ_k and $J_k (k = 1, 2, 3, ...)$ are the conserved densities and conserved fluxes, respectively.

Finally let us present the conservation laws for some particular cases:
i) If $\alpha = 1, \beta = 0$ in the equations (24)-(33) than we can receive conservation laws for SMBS

$$\begin{aligned}
\rho_1 &= -\frac{i}{2}\delta|q|^2, \\
\rho_2 &= -\frac{\delta}{2}(\frac{1}{2}q_x^*q + i\omega|q|^2), \\
\rho_3 &= -\frac{i\delta}{2}(-q_{xx}^*q + (|q|^2)^2 - \omega q_x^* q),
\end{aligned}$$

and

$$\begin{aligned}
J_1 &= (i\delta|q|^2\omega + i\eta) + (2\omega + iq_x)g_1 + 2g_2, \\
J_2 &= (iq_x\omega\frac{1}{q} - \frac{ip}{q})g_1 + (2\omega + iq_x)g_2 + 2g_3, \\
J_3 &= (iq_x\omega\frac{1}{q} - \frac{ip}{q})g_2 + (2\omega + iq_x)g_3 + 2g_4.
\end{aligned}$$

ii) If $\alpha = 0, \beta = 1$ in the equations (24)-(33) than we can get the conservation laws for cmKdVMB equations

$$\begin{aligned}
\rho_1 &= -\frac{i}{2}\delta|q|^2, \\
\rho_2 &= -\frac{\delta}{2}(\frac{1}{2}q_x^*q + i\omega|q|^2), \\
\rho_3 &= -\frac{i\delta}{2}(-q_{xx}^*q + (|q|^2)^2 - \omega q_x^* q),
\end{aligned}$$

and

$$\begin{aligned}
J_1 &= (\delta(q_x^*q - q^*q_x)\omega + i\eta) + (2i\omega\frac{q_x}{q} - q_{xx} - 2\delta|q|^2q)g_1 + (4\omega + 2i\frac{q_x}{q})g_2, \\
J_2 &= (-(q_{xx} + 2\delta|q|^2q)\omega\frac{1}{q} - \frac{ip}{q})g_1 + (2i\omega\frac{q_x}{q} - q_{xx} - 2\delta|q|^2q)g_2 + (4\omega + 2i\frac{q_x}{q})g_3, \\
J_3 &= (-(q_{xx} + 2\delta|q|^2q)\omega\frac{1}{q} - \frac{ip}{q})g_2 + (2i\omega\frac{q_x}{q} - q_{xx} - 2\delta|q|^2q)g_3 + (4\omega + 2i\frac{q_x}{q})g_4.
\end{aligned}$$

4. Conclusion

In the present paper, we construct infinitely many conservation laws for certain nonlinear evolution equations such as Hirota-Maxwell-Bloch system, Schrodinger-Maxwell-Bloch system, complex modified Korteweg de Vries-Maxwell-Bloch equations. The conservation laws were received from the Riccati form of the Lax pair. The existence of infinitely many conservation laws helpfully indicates the completely integrable property of these equations.

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References

[1] Ablowitz K J, Kaup D J, Newell A C, Segur H 1973 it Phys. Rev. Lett. **31** 125
[2] Kaup D J, Newell A C 1978 *J. Math. Phys.* **19** 798
[3] Wadati M, Konno K, Ichikawa Y H. 1979 *J. Phys. Soc. Jpn.* **46**

[4] Degasperis A. 1982 *Lett. Nuovo Cimento* **33** 425
[5] Fokas A. S. 1987 *Stud. Appl. Math* **77** 253
[6] Hereman W 2006 *Int. J. Quantum Chem* **106** 278
[7] Hereman W, Colagrosso M, Sayers R, Ringler A, Deconinck B, Nivala M, Hickman M. 2005 *In Differential Equations with Symbolic Computation* Basel
[8] Anderson I M. 2004 *The Variational Bicomplex D*, 318 pages
[9] Krasilshchik I S, Vinogradov A M 1998 *AMS: Providence*, Rhode Island
[10] Olver P J 1993 (Springer Verlag, New York)
[11] Lu X, Peng M 2013 *Commun Nonlinear Sci Numer Simulat* **18** 23042312
[12] Gao Zh, Gao Y, Su Ch, Wang Q and Mao B 2016 *Naturforsch* **71(1)a** 9-20
[13] Wang P, Tian B, Liu W, Qu Q, Li M and Sun K 2011 *Eur. Phys. J. D* **61**, 701-708
[14] Zhang H, Tian B, Meng X, Liu X, and Liu W 2009 *Eur. Phys. J. B* **72**, 233-239
[15] Guo R, Tian B, Lu X, Zhang H, and Liu W 2012 *Computational Mathematics and Mathematical Physics* **52** No. 4 565-577
[16] Yesmakanova K R, Shaikhova G N , Bekova G T, and Myrzakulova Zh R 2016 *Advances in Intelligent Systems and Computing* **441** 183-198
[17] Myrzakulov R, Mamyrbekova G K, Nugmanova G N, Lakshmanan M 2015 *Symmetry* **7(3)** 1352-1375
[18] Yesmakanova K R, Shaikhova G N , Bekova G T and Myrzakulov R 2016 *Journal of Physics: Conference Series* **738** 012018
[19] Yesmakanova K R, Shaikhova G N, Bekova G T 2016 *AIP Conference Proceedings* **1759** 020147
[20] Porsezian K and Nakkeeran K 1995 *Phys. Rev. Lett.* **74**, 2941
[21] Li Ch, He J *arXiv:1210.2501*
[22] Yang J, Li Ch, Li T, Cheng Zh 2013 *Chinese Physics Letters* **30**, 104201
[23] Li Ch, He J, Porsezian K 2013 *Physical Review E* **87** 012913
[24] Hisakado M and Wadati M 1995 *J. Phys. Soc. Jpn.* **64** 408